String Tension and Thermodynamics with Tree Level and Tadpole Improved Actions

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ABSTRACT

We calculate the string tension, deconfinement transition temperature and bulk thermodynamic quantities of the $SU(3)$ gauge theory using tree level and tadpole improved actions. Finite temperature calculations have been performed on lattices with temporal extent $N_\tau = 3$ and 4. Compared to calculations with the standard Wilson action on this size lattices we observe a drastic reduction of the cut-off dependence of bulk thermodynamic observables at high temperatures. In order to test the influence of improvement on long-distance observables at T_c we determine the ratio $T_c/\sqrt{\sigma}$. For all actions, including the standard Wilson action, we find results which differ only little from each other. We do, however, observe an improved asymptotic scaling behaviour for the tadpole improved action compared to the Wilson and tree level improved actions.

1   Introduction

Tree level and tadpole improved actions have been shown to yield a substantial reduction of cut-off dependences in the calculation of thermodynamic properties of the $SU(3)$ gauge theory [1, 2]. In the high (infinite) temperature limit this is quite evident already from a perturbative calculation of the pressure (p) or energy density (ϵ). In this limit high momentum modes give the dominant contribution to these observables. An improvement of the discretization scheme for the Euclidean action at short distances will thus naturally lead to a better representation of the ideal gas, Stefan-Boltzmann law on lattices with finite temporal extent N_τ. However, even at T_c the improved actions lead to a reduced cut-off dependence for some observables. Calculations of the surface tension or the latent heat of the first order deconfinement transition, for instance, show a strong reduction of the cut-off dependence on lattices with temporal extent $N_\tau = 3$ and 4. In this case it also has been found that the remaining cut-off dependence on coarse lattices is weaker for a tadpole improved action than for a tree-level improved action [2].

At high temperature the variation of p/T^4 or ϵ/T^4 with temperature is small. Both observables only slowly approach the ideal gas limit, which is consistent with the expectation that thermodynamics depends on a running coupling that varies logarithmically with temperature. The numerical simulations performed in a given discretization scheme at finite temperature are therefore particularly sensitive to a correct representation of the infinite temperature limit. An accurate determination of the temperature scale itself is, however, not needed for the observation of the dramatic improvement in approaching the continuum Stefan-Boltzmann limit at infinite temperature. This has been utilized in the calculations presented in [1] where the temperature scale has been fixed using an effective coupling scheme combined with the asymptotic form of the SU(3) β-function [3]. At temperatures close to the deconfinement transition, however, thermodynamic observables like p/T^4 vary rapidly. A comparison of the improvement achieved with different actions in this temperature regime requires an accurate determination of the temperature scale. For some fixed point actions this has been done by determining the critical temperature on lattices with different temporal extent [4]. Using calculations of T_c to set the scale has, however, the disadvantage that T/T_c gets to be known only at a few discrete values. We will use here the string tension to define a continuous temperature scale, $T/\sqrt{\sigma}$, for thermodynamic observables. This also is needed for the determination of thermodynamic observables like the energy density, which require the knowledge of the $SU(3)$ β-function, i.e. the derivative of the bare coupling with respect to the cut-off in the non-asymptotic regime.

In this paper we analyze the heavy quark potential with tree level and tadpole improved actions. From the potential the string tension is extracted in order to define a temperature scale, $T/\sqrt{\sigma}$. This in turn is used in a study of the thermodynamics of the SU(3) gauge theory. In the next section we discuss our calculations

β	$1 - \langle W^{1,1} \rangle$	u_0^2
3.80	0.463838(182)	0.681057
4.00	0.511237(275)	0.715008
4.20	0.560142(238)	0.748427
4.50	0.613736(130)	0.783413
5.00	0.664764(114)	0.815330
6.00	0.730540(74)	0.854716

Table 1: Tadpole improvement factors for the action $S^{(1,2)}$ defined in Eq. (2.1).

of the heavy quark potential and the determination of the string tension. The application of the resulting β-function for thermodynamic calculations is presented in Section 3. Section 4 contains our conclusions.

2 String Tension

In our calculations we use two different improved actions for the SU(3) gauge theory. These actions include in addition to the standard 1×1 Wilson loop an additional 1×2 or 2×2 loop, respectively,

$$
\begin{aligned}
S^{(1,2)} &= \sum_{x,\nu>\mu} \left(\frac{5}{3} W_{\mu,\nu}^{1,1}(x) - \frac{1}{6u_0^2} W_{\mu,\nu}^{1,2}(x) \right) \ , \\
S^{(2,2)} &= \sum_{x,\nu>\mu} \left(\frac{4}{3} W_{\mu,\nu}^{1,1}(x) - \frac{1}{48u_0^4} W_{\mu,\nu}^{2,2}(x) \right) \ .
\end{aligned}
\tag{2.1}
$$

Here u_0 denotes a tadpole improvement factor which we have chosen to be defined through the plaquette expectation value [5], i.e. $u_0 \equiv (1 - \langle W_{\mu,\nu}^{1,1}(x) \rangle)^{1/4}$. The tree level improved actions are obtained for $u_0 \equiv 1$. The high temperature ideal gas limit for these actions has been analyzed previously [1] and first calculations of the pressure using the action $S^{(2,2)}$ have been presented there. The analysis of the infinite temperature ideal gas limit suggests, in fact, that the (1,2)-action is superior to the (2,2)-action. Although the leading $\mathcal{O}(a^2)$ corrections are eliminated in both cases, it turns out that the remaining higher order contributions are much smaller for the (1,2)-action. In order to calculate the string tension from the long distance part of the heavy quark potential we have performed simulations of the $SU(3)$ gauge theory on lattices of size 16^4 and 24^4 at several values of the gauge coupling $\beta = 6/g^2$. The heavy quark potential has been extracted from smeared Wilson loops following closely the smearing approach described in [6], i.e. Wilson loops are constructed

from smeared links which are obtained by iterating the replacement process

$$U_\mu(x) \to U_\mu(x) + \gamma \sum_{\nu \neq \mu} U_\nu(x) U_\mu(x + \hat{\nu}) U_\nu^\dagger(x + \hat{\mu}) \qquad (2.2)$$

several times. Some tests have been performed to find optimal values for the smearing parameter γ for the range of couplings explored here. The value γ varies between 0.2 and 0.8. This allows to achieve a reasonable large overlap with the ground state already after about 10 to 15 smearing steps[a]. The potential at distance R is then determined from the asymptotic behaviour of smeared Wilson loops, $W(R, L)$,

$$V(R) = \lim_{L \to \infty} \ln\left(\frac{W(R, L)}{W(R, L + 1)}\right) \quad . \qquad (2.3)$$

Typically, for each value of the gauge coupling we have analyzed Wilson loops on 300 configurations which were separated by 10 updates performed with an over-relaxed heat bath algorithm (1 update $\equiv$ 4 over-relaxation steps followed by 1 heat bath step). In the case of the (1,2)-action we also investigated the effect of tadpole improvement. For this purpose we have first determined self-consistently the factor u_0 at a few values of the gauge coupling. These numbers are given in Table 1. Spline interpolations of these values have then been used for simulations at other values of the gauge coupling. Calculations of Wilson loops have been performed at 10 to 15 values of the gauge coupling. We generally observe that the ratios of smeared Wilson loops become, within errors, independent of L already for rather small values, i.e. for $L > L_{\min} \simeq (2 - 5)$ for large R. The potential has then been obtained from a weighted average of the logarithm of ratios $W(R, L)/W(R, L + 1)$ with $L > L_{\min}$. These calculations have been performed on lattices of size 16^4, except for the largest values of the gauge coupling where calculations have been performed on a 24^4 lattice. Similar results have been obtained with the tree level improved action $S^{(2,2)}$ on lattices of size 24^4. The potentials have then been fitted to the ansatz

$$V(R) = V_0 + \frac{\hat{\alpha}}{R} + \hat{\sigma} R \quad , \qquad (2.4)$$

for distances $R > R_{\min}$ in order to extract the string tension. For large values of R, roughly $Ra > 1/4$fm [8], the coefficient of the Coulomb-like term is expected to be determined by string fluctuations, $\hat{\alpha} = -\pi/12$ [9]. Indeed, for β corresponding to a lattice spacing smaller than $a \approx 0.13$fm we find values for $\hat{\alpha}$ which are consistent with this (Table 2). Since we are interested in the long distance behaviour of the potential we followed the strategy to fix the coefficient of the Coulomb-like term for the determination of the string tension. The minimal value of $R_{\min}$ was chosen to be the maximum of $Ra = 0.25$fm and $R = 2\sqrt{2}$, so that distortions of rotational symmetry which are present at small distances are not large. This approach was compared to a systematic increase of $R_{\min}$ until stable results have been obtained for $\hat{\sigma}$. Typically this was the case for $R_{\min}^* \gtrsim 1/(2\sqrt{\hat{\sigma}})$ i.e. $R_{\min}^* a \gtrsim 0.25$fm. We then have

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[a]Details on the optimization of our smearing procedure can be found in Ref. [7].

Tree-level improved (1,2)-action				
$N_\sigma^3 N_\tau$	β	$R_{0.25\text{fm}}$	$R_{\min}^\star$	$\hat{\alpha}$
16^4	4.300	1.98	2.828	-0.318 (82)
16^4	4.400	2.34	2.828	-0.335 (68)
16^4	4.600	2.97	3.0	-0.290 (36)
16^4	4.800	4.11	4.0	-0.229 (76)
24^4	5.000	4.79	5.0	-0.284 (30)
Tadpole improved (1,2)-action				
$N_\sigma^3 N_\tau$	β	$R_{0.25\text{fm}}$	$R_{\min}^\star$	$\hat{\alpha}$
16^4	4.600	1.96	2.828	-0.310 (61)
16^4	4.800	2.53	2.828	-0.262 (42)
24^4	5.100	3.74	4.0	-0.288 (40)
24^4	5.250	4.13	4.24	-0.316 (112)
24^4	5.400	4.76	5.0	-0.212 (54)
Tree level improved (2,2)-action				
$N_\sigma^3 N_\tau$	β	$R_{0.25\text{fm}}$	$R_{\min}^\star$	$\hat{\alpha}$
24^4	4.600	1.98	2.828	-0.299 (45)
24^4	4.800	2.76	2.828	-0.296 (21)
24^4	5.000	3.70	4.0	-0.274 (22)
24^4	5.200	4.85	5.0	-0.271 (47)
24^4	5.400	6.43	5.0	-0.289 (29)
24^4	5.600	8.10	5.0	-0.266 (22)

Table 2: Coefficient of the Coulomb-like term obtained from 3 parameter fits to the heavy quark potential which have been calculated with tree level and tadpole improved (1,2)-actions and tree level improved (2,2) action. $R_{0.25\text{fm}}$ is the distance in lattice units corresponding to a physical value of 0.25 fm and $R_{\min}^*$ is the minimal distance used in the fits to the potential for the determination of $\hat{\alpha}$. The values of alpha and the errors have been calculated in the same way as for the string tension according to the formula (2.5)
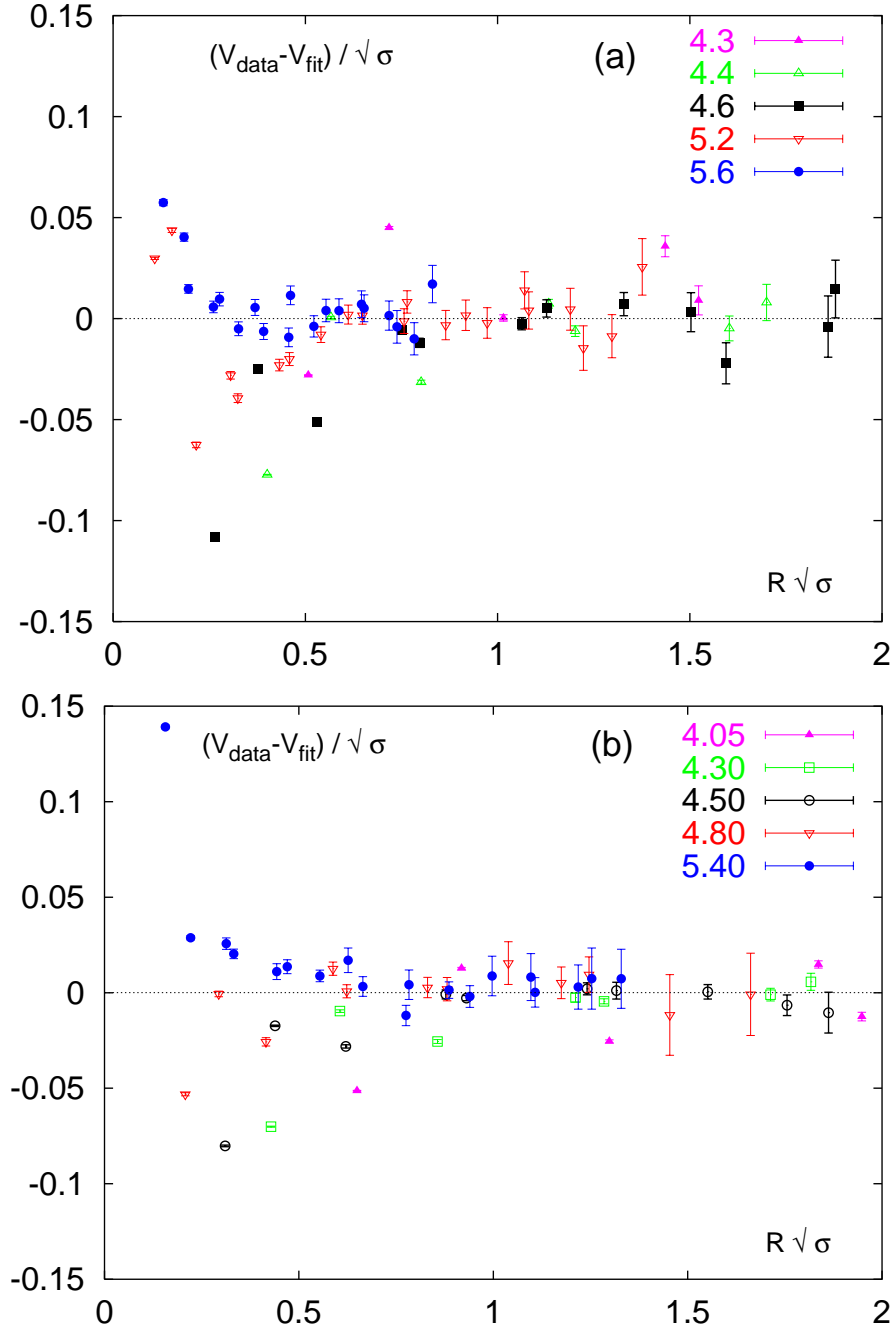
Figure 1: The difference between the calculated potentials and the best fit obtained with fixed $\hat{\alpha}$ in units of $\sqrt{\sigma}$ versus $R\sqrt{\sigma}$. Shown are results for the tree level improved (2,2)-action (a) and the tadpole improved (1,2)-action (b) for different values of the gauge coupling. Tree level results for the latter action look similar.

further increased $R_{\min}$ and have averaged the results for $\hat{\sigma}$ obtained from several fits with $R_{\min} > R^*_{\min}$ in order to minimize remaining distortion effects resulting from missing rotational invariance of the potential at these distances. Since the results for $\hat{\sigma}$ at distances larger than $R^*_{\min}$ agree within errors we calculated the mean value according to the fit formula of a constant, but modified the error formula by introducing a factor $\sqrt{N}$ which takes into account that the different values of $\hat{\sigma}$ are strongly correlated.

$$
\hat{\sigma} = \left(\sum_{\hat{R}_{\min} \geq \hat{R}^*_{\min}} \frac{\hat{\sigma}(\hat{R}_{\min})}{(\Delta\hat{\sigma}(\hat{R}_{\min}))^2} \right) \left(\sum_{\hat{R}_{\min} \geq \hat{R}^*_{\min}} \frac{1}{(\Delta\hat{\sigma}(\hat{R}_{\min}))^2} \right)^{-1}
$$

$$
\Delta\hat{\sigma} = \left(\frac{1}{N} \sum_{\hat{R}_{\min} \geq \hat{R}^*_{\min}} \frac{1}{(\Delta\hat{\sigma}(\hat{R}_{\min}))^2} \right)^{-\frac{1}{2}}. \tag{2.5}
$$

The resulting values for the string tension for the different actions are given in Table 3 and in Table 4. Also given there are the minimal values $L_{\min}$ and $R^*_{\min}$ used to extract and fit the potential for a given value of β. In the last column we show the values of $\hat{\sigma}$ obtained from the 3 parameter fit of the potential (Eq. 2.4). Except for $\beta = 4.400$ for the tree level improved (1,2) and $\beta = 4.500$ for the tree level improved (2,2) action all values of the string tensions obtained from 2 parameter fits agree within errors with the ones from 3 parameter fits. The χ^2 of the fits is comparable.

In Figure 1 we show the difference between the calculated potential and the best fit obtained with $\hat{\alpha} = -\pi/12$. Results are shown for the tree level improved (2,2)-action and the tadpole improved (1,2)-action. All fits have been performed for distances $R\sqrt{\hat{\sigma}} \gtrsim 0.5$. We note that the agreement between the numerical data and the fit generally remains good at distances smaller than the fitting interval. For smaller values of β we observe, however, a scattering of the data for $R\sqrt{\hat{\sigma}} < 0.5$ due to the lack of rotational invariance. This effect becomes smaller for larger values of β, i.e. closer to the continuum limit. For large values of β the potential can also be obtained at shorter physical distances. Here we clearly observe that the potential at small distances would prefer a larger coupling for the Coulomb term than used in our fit. This is reflected in the increase of $V_{\text{data}} - V_{\text{fit}}$ at short distances.

2.1 Deviations from asymptotic scaling

Improved actions depend on several couplings for the different Wilson loops appearing in the action. Through the specific choice of these couplings a particular trajectory in a multi-dimensional parameter space is defined on which the continuum limit is approached. Although the improvement scheme for the actions does not aim at improving the approach to asymptotic scaling we may test in how far the scaling behaviour is modified through the specific choice of a trajectory. The results for the

Tree-level improved (1,2)-action					
$N_\sigma^3 N_\tau$	β	$L_{\min}$	$R^\star_{\min}$	$\hat{\sigma}_{2\text{par}}$	$\hat{\sigma}_{3\text{par}}$
16^4	3.850	1	2.828	0.35173 (328)	0.35789 (1327)
16^4	4.000	2	2.828	0.19923 (162)	0.19748 (836)
16^4	4.040	2	2.828	0.17542 (104)	0.17288 (457)
16^4	4.100	3	2.828	0.14234 (165)	0.13871 (842)
16^4	4.150	3	2.828	0.11423 (131)	0.11079 (586)
16^4	4.300	4	2.828	0.07265 (107)	0.06987 (489)
16^4	4.400	4	2.828	0.05412 (51)	0.05055 (182)
16^4	4.600	4	3.0	0.03201 (43)	0.03105 (182)
16^4	4.800	5	4.0	0.01634 (48)	0.01734 (257)
24^4	5.000	6	5.0	0.01188 (18)	0.01104 (84)
Tadpole improved (1,2)-action					
$N_\sigma^3 N_\tau$	β	$L_{\min}$	$R^\star_{\min}$	$\hat{\sigma}_{2\text{par}}$	$\hat{\sigma}_{3\text{par}}$
16^4	4.050	1	2.828	0.42159 (217)	0.42216 (1462)
16^4	4.150	2	2.828	0.29927 (340)	0.32253 (3625)
16^4	4.185	2	2.828	0.27153 (216)	0.25530 (1645)
16^4	4.200	2	2.828	0.26205 (198)	0.25367 (1124)
16^4	4.250	2	2.828	0.21829 (176)	0.21355 (893)
16^4	4.300	2	2.828	0.18343 (103)	0.17852 (496)
16^4	4.350	3	2.828	0.15416 (238)	0.15947 (1265)
16^4	4.400	3	2.828	0.12994 (154)	0.12581 (754)
16^4	4.500	3	2.828	0.09633 (53)	0.09536 (252)
16^4	4.600	4	2.828	0.07154 (90)	0.06839 (366)
16^4	4.800	4	2.828	0.04315 (55)	0.04296 (235)
24^4	5.100	6	4.0	0.01967 (22)	0.01906 (87)
24^4	5.250	6	4.24	0.01604 (53)	0.01513 (233)
24^4	5.400	6	5.0	0.01227 (20)	0.01308 (97)

Table 3: String tension obtained from 2 and 3 parameter fits to the heavy quark potential which have been calculated with tree level and tadpole improved (1,2)-actions. $L_{\min}$ is the minimal extent of smeared Wilson loop $W(R, L)$ used to extract the potential at distance R. $R^*_{\min}$ is the minimal distance used in fits to the potential for the determination of the string tension.

Tree-level improved (2,2)-action				
β	$L_{\min}$	$R^{\star}_{\min}$	$\hat{\sigma}_{2\mathrm{par}}$	$\hat{\sigma}_{3\mathrm{par}}$
4.300	3	2.828	0.24547 (981)	0.31077 (7679)
4.380	3	2.828	0.17401 (278)	0.15589 (1534)
4.396	3	2.828	0.16334 (228)	0.16059 (1366)
4.400	3	2.828	0.16052 (207)	0.16778 (1474)
4.406	3	2.828	0.15603 (186)	0.14942 (1257)
4.430	3	2.828	0.13882 (141)	0.13238 (900)
4.450	4	2.828	0.12787 (102)	0.12698 (1949)
4.500	4	2.828	0.10422 (53)	0.08755 (867)
4.550	4	2.828	0.08636 (83)	0.08431 (461)
4.600	4	2.828	0.07060 (48)	0.06957 (236)
4.800	4	2.828	0.03606 (13)	0.03536 (57)
5.000	4	4.0	0.02046 (13)	0.02021 (48)
5.200	5	5.0	0.01169 (16)	0.01166 (76)
5.400	5	7.0(5.0)	0.00669 (14)	0.00642 (46)
5.600	5	8.0(5.0)	0.00426 (14)	0.00425 (35)

Table 4: String tension obtained from 2 and 3 parameter fits to the heavy quark potential which have been calculated with the tree level improved (2,2)-action on lattices of size 24^4. $L_{\min}$ and $R^*_{\min}$ are explained in Table 3. For the last two β values we used a smaller value of $R^*_{\min} = 5.00$ in the 3 parameter fit to obtain stable results.
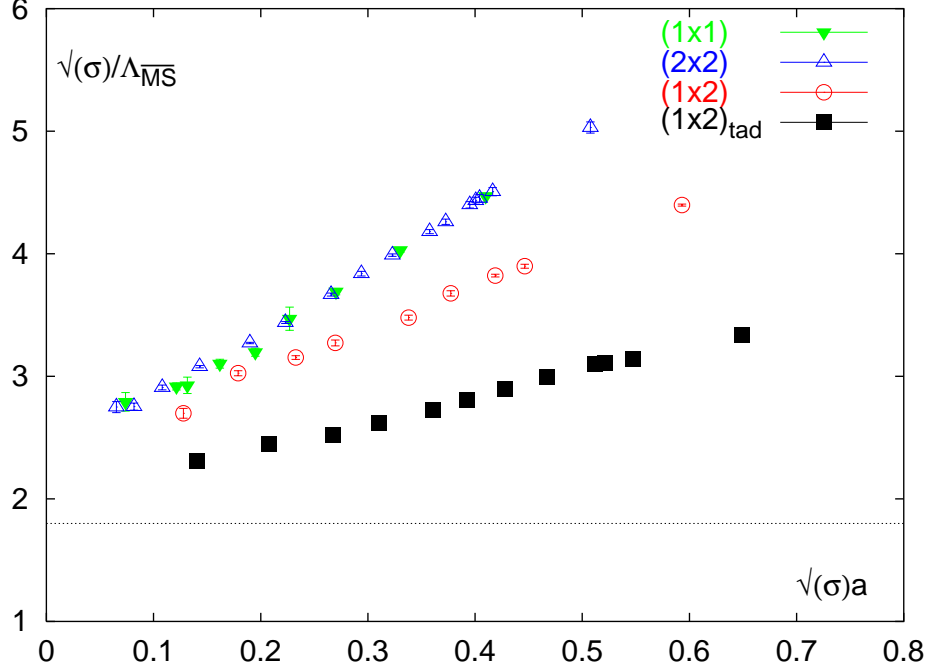
Figure 2: $\sqrt{\sigma}/\Lambda_{\overline{\text{MS}}}$ versus $\sqrt{\sigma}a$ for various actions. Shown are results for $\sqrt{\sigma}a/R(g^2)$, normalized to $\Lambda_{\overline{\text{MS}}}$. Here $R(g^2)$ denotes the 2-loop β-function. In the case of the (2,2)-action the ratio of $\Lambda_{\text{tree}}^{(2,2)}/\Lambda_{\overline{\text{MS}}}$ was not known to us and we have, therefore, used an arbitrary normalization factor of 0.15. The horizontal line indicates the value of the continuum extrapolation of $\sqrt{\sigma}/\Lambda_{\overline{\text{MS}}}$ for the Wilson action taken from Ref. [6].

string tension given in Table 3 and 4 can be used to analyze the relation between the bare gauge coupling and the lattice cut-off, $\beta(a) \equiv 6/g^2(a)$. In the case of the Wilson action this has been analyzed in quite some detail [10]. In a most straightforward way the deviations from asymptotic scaling become visible when one divides the results obtained for $\sqrt{\sigma}a$ by the universal 2-loop form of the β-function. This yields $\sqrt{\sigma}/\Lambda_L$. The perturbative expansion of the tadpole improved (1,2)-action coincides up to $\mathcal{O}(g^2)$ with that of the tree level improved (1,2)-action with a modified gauge coupling $\tilde{\beta} \equiv \beta(1 + \alpha_2 g^2/3)$, where $\alpha_2 = -0.366263(N^2 - 1)/4N$ denotes the $\mathcal{O}(g^2)$ expansion coefficient for the plaquette expectation value calculated with the tree level improved action [1]. From this we obtain the ratio of Λ-parameters for tree level and tadpole improved (1,2)-actions,

$$\frac{\Lambda_{\text{tad}}^{(1,2)}}{\Lambda_{\text{tree}}^{(1,2)}} = e^{\alpha_2/6b_0} \quad , \tag{2.6}$$

with $b_0 = 11N/48\pi^2$. Using also the ratios of lattice Λ-parameters to $\Lambda_{\overline{\text{MS}}}$ [11, 12] we obtain $\sqrt{\sigma}/\Lambda_{\overline{\text{MS}}}$ for different actions. This is shown in Figure 2. The numbers are plotted versus $\sqrt{\sigma}a$ in order to be able to compare results at the same value of the cut-off. The slope of these curves is an indication for the deviations from asymptotic

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scaling in the regime of couplings investigated. We note that the tree level improved (2,2)-action shows similar scaling violations as the Wilson action while these are reduced for the tree level and even more for the tadpole improved (1,2)-actions. A similar behaviour is also obtained from calculations of the critical temperature with the tree level improved (1,2) action [13].

3 Thermodynamics

3.1 $T_c/\sqrt{\sigma}$

The string tension calculated in the previous section can be used to fix the temperature scale $T/\sqrt{\sigma} = 1/(N_\tau \sqrt{\hat{\sigma}})$. In particular, we have calculated the ratio $T_c/\sqrt{\sigma}$ by determining the critical couplings for the different tree level and tadpole improved actions on lattices with temporal extent $N_\tau = 3$ and 4. Pseudo-critical couplings on finite spatial lattices have been determined from the location of the peak in the Polyakov loop susceptibility [3]. For the tree level and tadpole improved (1,2)-actions we, furthermore, have performed a detailed study of the finite volume dependence of the critical couplings on lattices with temporal extent $N_\tau = 4$ and $N_\sigma = 16$, 24 and 32 [2]. In these cases the critical couplings have been extrapolated to the infinite volume limit using the ansatz

$$\beta_c(N_\tau, N_\sigma) = \beta_c(N_\tau, \infty) - h\left(\frac{N_\tau}{N_\sigma}\right)^3 \tag{3.1}$$

which is appropriate for first order phase transitions. We note that in the continuum limit the difference $\beta_c(N_\tau, \infty) - \beta_c(N_\tau, N_\sigma)$ is proportional to the finite volume shift in the critical temperature, i.e. $(T_{c,\infty} - T_{c,V})/T_{c,\infty}$. In this limit Eq. (3.1) thus characterizes a physical property of QCD in a finite volume, the finite size dependence of peaks in susceptibilities which develop into a singularity in the infinite volume limit. Eq. (3.1) then reads

$$\frac{T_{c,\infty} - T_{c,V}}{T_{c,\infty}} = \frac{4\pi^2}{33} \frac{h}{VT^3} \tag{3.2}$$

The parameter h thus is independent of the lattice action up to finite cut-off corrections. Given the current uncertainties in the numerical values of h these finite cut-off corrections can, however, not be disentangled from the statistical errors.
We find for the tree level and tadpole improved (1,2)-actions proportionality factors h which are consistent with each other as well as with earlier results obtained with

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other actions,

$$
h = \begin{cases}
0.082 \ (32) & \text{(1,1) Wilson action [15], } N_\tau = 4 \\
0.072 \ (77) & \text{(1,1) Wilson action [15], } N_\tau = 6 \\
0.101 \ (34) & \text{(1,2) tree level action, } N_\tau = 4 \\
0.068 \ (45) & \text{(1,2) tadpole action, } N_\tau = 4 \\
0.122 \ (54) & \text{RG action [18], } N_\tau = 3 \\
0.133 \ (63) & \text{RG action [18], } N_\tau = 4
\end{cases}
\tag{3.3}
$$

The calculated critical couplings as well as the extrapolations to the infinite volume limit are summarized in Table 5 for the different actions. We note that our results for the tree level improved (1,2)-action are consistent with those of Ref. [13]. In Table 5 we also give results from [13] for larger values of N_τ and use the ansatz of Eq. (3.1) to extrapolate to the critical couplings on an infinite lattice. For the constant h we use in these cases the weighted average of the values given in Eq. (3.3), *i.e.* $h = 0.093 \ (18)$.

In order to extract the critical temperature in units of the square root of the string tension we determine $\hat{\sigma}$ at $\beta_c(N_\tau, \infty)$ from an interpolation with an exponential ansatz, $\hat{\sigma} = A \exp(-\beta/2Nb_0 + f(\beta))$, where $f(\beta)$ is a third order polynomial in β^{-1}. The results for $T_c/\sqrt{\sigma}$ are shown in Table 5 and in Figure 3.
The Wilson action $\beta_c(N_{\tau,\infty})$ for $N_\tau =4$ and 6 are taken from [15]. For $N_\tau = 8$ and $N_\tau = 12$ we performed the infinite volume extrapolation based on Eq. (3.1) with $h = 0.093$ by using β_c from [3]. Recently there has been a new measurement of the string tension for the Wilson gauge action by Edwards et al. [14]. We used their "best" parameterization of the string tension to determine $\sqrt{\hat{\sigma}}$ at β_c. The results are displayed in Table 6 and Figure 3. The strong cut-off effect so far reported for $T_c/\sqrt{\sigma}$ is no longer visible in these new results [b]. For the continuum extrapolation we find the following value

$$
\frac{T_c}{\sqrt{\sigma}} = 0.630 \pm 0.005 \quad ,
\tag{3.4}
$$

which is consistent with all ratios $T_c/\sqrt{\sigma}$ extracted by us with improved actions on $N_\tau \geq 4$ lattices (see Table 5). In Figure 3 and Table 7 we give the results obtained from calculations with different actions. This also includes results obtained with a renormalization group improved action [18]. The latter does lead to a slightly larger value for $T_c/\sqrt{\sigma}$. However, also in this case the cut-off dependence is comparable to the other actions. This suggests that the difference between the results obtained with the RG-improved action in Ref. [18] and the results presented here is mainly due to differences in the analysis of the heavy quark potential rather than due to differences in the improvement scheme. In general we find that the cut-off dependence in the

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[b]Previous estimates of $T_c/\sqrt{\sigma}$ for the Wilson action at low β were based on preliminary results for the string tension [16] which propagated through the literature. Published values [17] and recent high statistics results [14] for the string tension are lower than the preliminary ones and thus lead to the increase in $T_c/\sqrt{\sigma}$ at $N_\tau = 4$ and 6.

Tree level improved (1,2)-action			
N_σ^3	N_τ	β_c	$T_c/\sqrt{\sigma}$
12^3	3	3.9079 (6)	
$(\infty)^3$	3	3.9094 (6)(3)	0.630 (5)
32^3	4	4.0729 (3)	
$(\infty)^3$	4	4.0730 (3)	0.636 (4)
20^3	5	4.19963 (14)	
$(\infty)^3$	5	4.20108 (14)(28)	0.631 (5)
24^3	6	4.31466 (24)	
$(\infty)^3$	6	4.31611 (24)(28)	0.632 (5)
Tree level improved (2,2)-action			
N_σ^3	N_τ	β_c	$T_c/\sqrt{\sigma}$
24^3	4	4.3995 (2)	
$(\infty)^3$	4	4.3999 (2)(1)	0.625 (4)
Tadpole improved (1,2)-action			
N_σ^3	N_τ	β_c	$T_c/\sqrt{\sigma}$
12^3	3	4.1868 (4)	
$(\infty)^3$	3	4.1882 (4)(3)	0.643 (3)
32^3	4	4.3522 (4)	
$(\infty)^3$	4	4.3523 (4)	0.639 (6)

Table 5: Critical couplings for the tree level and tadpole improved actions. In each case we give the result for the largest spatial lattice on which simulations have been performed and the infinite volume extrapolation. Details on the determination of critical couplings for the (1,2)-actions for $N_\tau = 3$ and 4 on lattices with different spatial extent are given in [2]. The finite lattice results for $N_\tau = 5$ and 6 are taken from [13]. Infinite volume extrapolations are based on Eq. (3.1) with $h = 0.093$, except for the case of the $N_\tau = 4$ (1,2)-actions where a detailed finite volume scaling analysis has been performed in [2]. The second error on β_c in the infinite volume limit is the systematic error due to the error on h. In the last column we also give the ratio $T_c/\sqrt{\sigma}$. The errors on $T_c/\sqrt{\sigma}$ due to the error on β_c and on $\sqrt{\hat{\sigma}}$ have been added.

Wilson action			
N_σ^3	N_τ	β_c	$T_c/\sqrt{\sigma}$
$(\infty)^3$	4	5.6925 (2)	0.627 (6)
$(\infty)^3$	6	5.8941 (5)	0.632 (11)
$(\infty)^3$	8	6.0624 (9) (3)	0.629 (6)
$(\infty)^3$	12	6.3380 (13) (10)	0.630 (5)

Table 6: Critical couplings for the Wilson gauge action. We give the results for the infinite volume extrapolations for $N_\tau = 4$ and 6 from [15] and for $N_\tau = 8$ and 12 based on Eq. (3.1) with $h = 0.093$, where the values for β_c on finite lattices are taken from [3]. The second error given is the systematic error due to the uncertainty in the extrapolation. In the last column we give the ratio $T_c/\sqrt{\sigma}$ using the string tension parameterization given in [14].
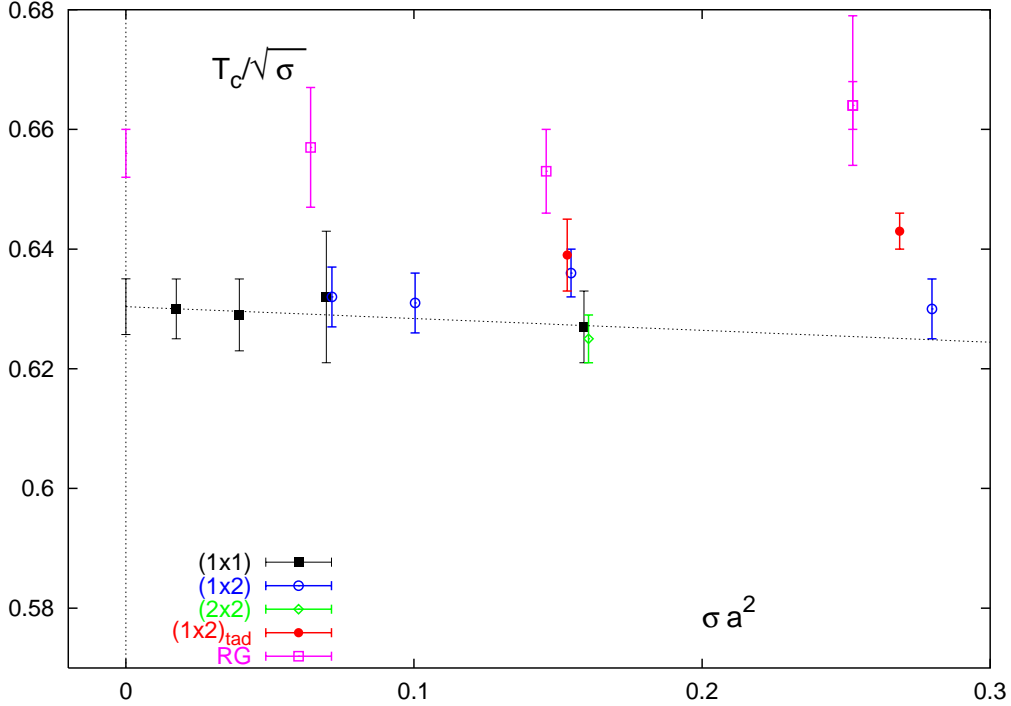

Figure 3: The critical temperature in units of the square root of the string tension for various actions versus the square of the cut-off. The $N_\tau = 6$ point of the tree level (1,2) improved action has been slightly shifted to make it distinguishable from the Wilson action point.

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ratio $T_c/\sqrt{\sigma}$ is quite small for all actions. We also note that the results on $N_\tau = 3$ lattices are in good agreement with those obtained on larger lattices. This is quite different from the surface tension analysis [2]. It may, however, indicate that the latter is more sensitive to high momentum modes than the ratio $T_c/\sqrt{\sigma}$, which is also reflected in the fact that the Wilson action does not show large finite cut-off effects in $T_c/\sqrt{\sigma}$ whereas there are large finite cut-off dependences in the surface tension.

action	β_c	$T_c/\sqrt{\sigma}$
standard Wilson	5.69254 (24)	0.627 (6)
(2,2) (tree level improved)	4.3999 (3)	0.625 (4)
(1,2) (tree level improved)	4.0730 (3)	0.636 (4)
(1,2) (tadpole improved)	4.3523 (4)	0.639 (6)
(1,2) (RG improved)	2.2879 (11)	0.653 (6)(1)

Table 7: Critical temperature in units of $\sqrt{\sigma}$ on lattices with temporal extent $N_\tau = 4$. Infinite volume extrapolations for the critical couplings have been performed in all cases. Further details on the data for the RG-improved action can be found in [18].

3.2 Pressure of the SU(3) gauge theory

The temperature dependence of the pressure can be obtained from an integration of action densities, $\langle S \rangle$, calculated at zero and non-zero temperature, respectively[c],

$$\left. \frac{p}{T^4} \right|_{\beta_0}^{\beta} = N_\tau^4 \int_{\beta_0}^{\beta} \mathrm{d}\beta' (\langle \tilde{S} \rangle_0 - \langle \tilde{S} \rangle_T) \quad . \tag{3.5}$$

Here β_0 is a value of the coupling constant below the phase transition point at which the pressure can safely be approximated by zero. The subscripts 0 and T refer to calculations of the action expectation values on the zero temperature and non-zero temperature lattices, respectively. We also note that in the case of tadpole improved actions the action itself depends implicitly on the gauge coupling through the tadpole factor $u_0(\beta)$. This has to be taken into account in the calculation of derivatives with respect to β. We therefore have introduced in Eq. (3.5) the quantity $\tilde{S}$ defined as

$$\tilde{S} = S - \beta \frac{\mathrm{d}S}{\mathrm{d}\beta} \quad . \tag{3.6}$$

Using the string tension values given in Table 3 and 4 and normalizing these to the string tension at β_c, we obtain the temperature in units of T_c. The temperature

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[c]We refer to Refs. [3, 19] for more details on the formalism.

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at intermediate values of the coupling has been obtained from an interpolation. With this we can reanalyze also the results for the pressure obtained for the tree level improved (2,2)-action given in [1]. It simply amounts to a modification of the temperature scale, i.e. the ordinate of Figure 3 in [1]. While this has practically no consequences at high temperature, it leads to visible shifts close to T_c. In Figure 4 we show the results of calculations with improved actions on $16^3 \times 4$ lattices. These are compared to the continuum extrapolation obtained from calculations with the Wilson action on lattices with temporal extent $N_\tau = 4$, 6 and 8 [3]. We find that all improved action calculations are close to the continuum extrapolation, while the standard Wilson calculation on a $N_\tau = 4$ lattice clearly deviates substantially.

We furthermore have calculated the pressure for the tree level and tadpole improved (1,2)-actions on a lattice of size $12^3 \times 3$. These results are shown in Figure 4b. Here we also show the result of a calculation using a fixed point action for $N_\tau = 3$ [4]. These are in good agreement with the continuum extrapolation obtained from the Wilson action and seem to be closer to the result obtained with the tadpole improved action than the tree level result. For temperatures in the range $(2 - 3)T_c$ the tadpole action yields about 10% smaller values for the pressure than the tree level action, although both approach the same infinite temperature limit. We also note that we do not observe any significant cut-off dependence when comparing calculations on lattices with temporal extent $N_\tau = 3$ and 4 despite the fact that the cut-off dependence in the infinite temperature limit leads to about 15% differences in the Stefan-Boltzmann limit. This is, to some extent, in accordance with the analysis of the Wilson action, where we noted already that the cut-off distortion in this temperature range is only half as large as expected on the basis of calculations for the ideal gas limit [3]. It seems that these are further reduced in calculations with improved actions. Using the temperature scale defined by the string tension calculations we also can extract the β-function, $ad\beta/da$, outside the validity regime of the asymptotic 2-loop form, i.e. in the coupling range explored here. With this, further thermodynamic observables can be extracted. In particular, we obtain for the difference between the energy density, ϵ, and $3p$,

$$\frac{\epsilon - 3p}{T^4} = \left(\frac{N_\tau}{N_\sigma}\right)^3 \left(a\frac{d\beta}{da}\right)\left[\langle\tilde{S}\rangle_0 - \langle\tilde{S}\rangle_T\right] \quad . \tag{3.7}$$

In Figure 5 we compare the result obtained with tree level and tadpole improved actions on lattices with temporal extent $N_\tau = 4$ with corresponding results obtained with the Wilson action for $N_\tau = 8$. In the case of the Wilson action it has actually been observed that results for $N_\tau = 6$ and 8 coincide within errors and may thus be taken as the continuum limit result [3]. The good agreement we find here with the Wilson action calculation confirms this observation, i.e. cut-off effects are indeed small in $(\epsilon - 3p)/T^4$. In fact, the strong cut-off effects present in the ideal gas limit cancel exactly in this quantity, which in the high temperature limit is $\mathcal{O}(g^4(T))$. The good agreement between results obtained with different actions also is an excellent consistency check for the determination of the temperature scales and the analysis
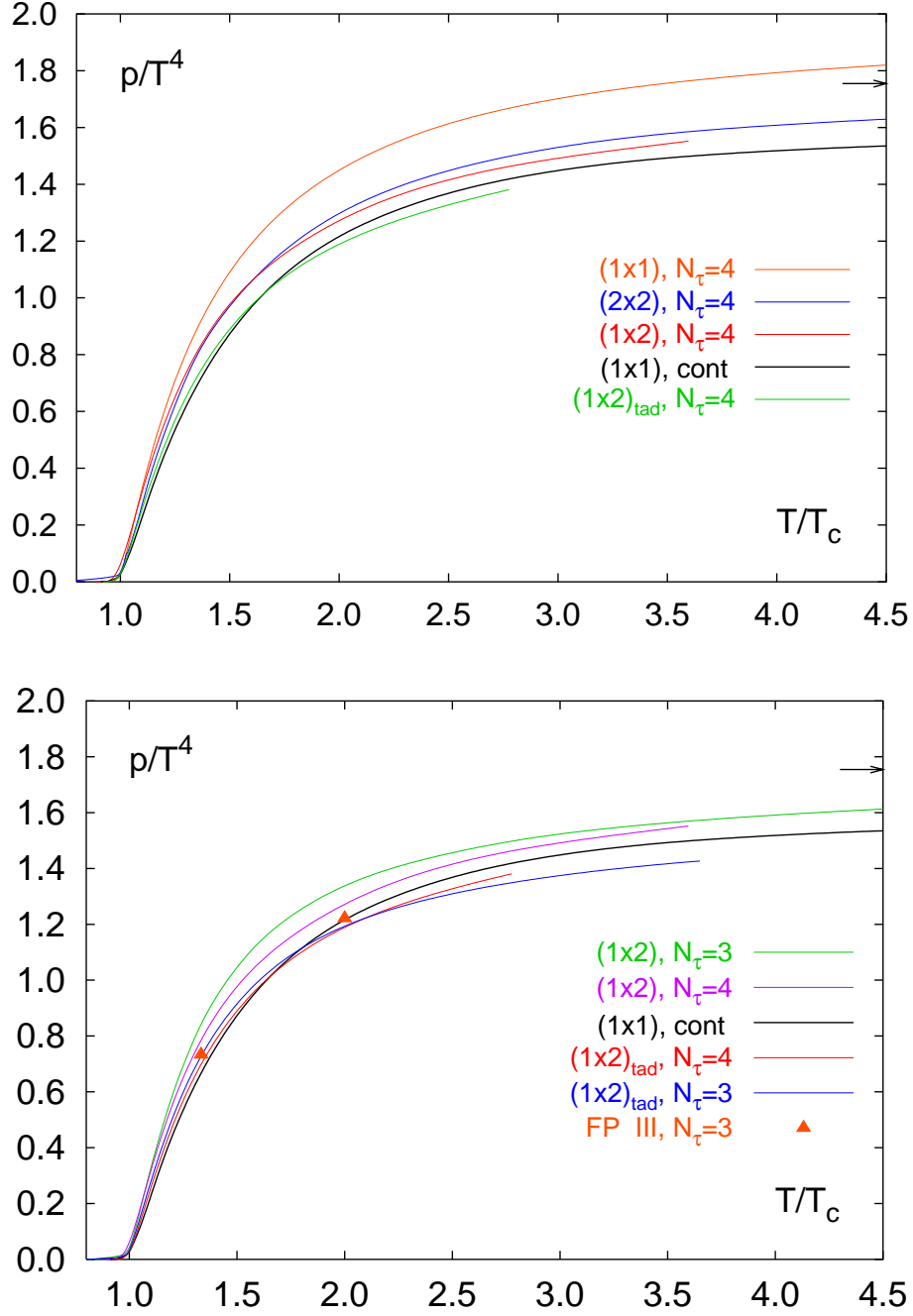
Figure 4: Pressure of the SU(3) gauge theory calculated with the Wilson action and different improved actions on $N_\tau = 4$ lattices (upper figure). The lower figure shows a comparison of calculations with tree level and tadpole improved actions on $N_\tau = 3$ and 4 lattices. Also shown there are results from a calculation with a fixed point action (triangles). The arrows indicate the ideal gas result in the continuum limit. For comparison we also show the continuum extrapolation obtained from calculations with the Wilson action [3].
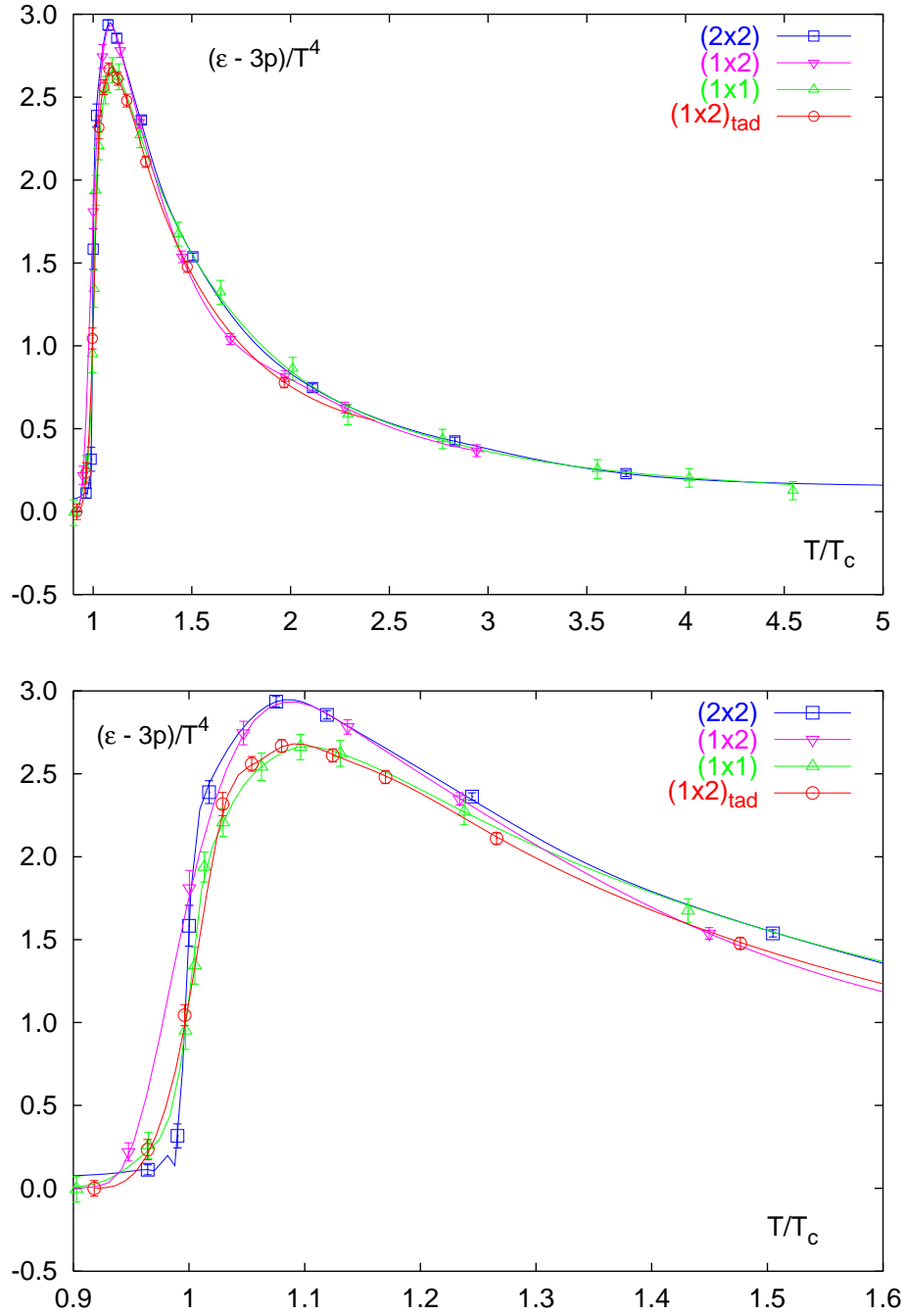
Figure 5: The difference $(\epsilon - 3p)/T^4$ versus T/T_c calculated on lattices with temporal extent $N_\tau = 4$ for various improved actions. The results are compared to a calculation with the Wilson action on a $N_\tau = 8$ lattice. In the lower figure the peak of $(\epsilon - 3p)/T^4$ is shown in detail.

performed here.

4   Conclusions

We have analyzed thermodynamic properties of the SU(3) gauge theory using tree level and tadpole improved actions. In general we find that the use of improved actions does lead to a significant reduction of the cut-off dependence in observables like the energy density, the pressure and even the surface tension at T_c. The improvement has, however, little effect on the calculation of long-distance quantities like $T_c/\sqrt{\sigma}$. On the other hand we do observe an improved scaling behaviour for the tadpole improved action. There also might be a slight advantage in the use of the tadpole improved action for the analysis of bulk thermodynamic observables like the pressure. However, at present this cannot be further quantified within the accuracy of our calculations.

It now seems that the systematic cut-off dependencies in calculations of thermodynamic observables (T_c, equation of state, latent heat, surface tension,..) are well under control with presently used improved actions. Remaining ultra-violet cut-off dependences, which without doubt still are present, are hidden by statistical errors and/or inaccuracies in the determination of the observables. The latter result, for instance, also from infra-red cut-off effects. This is true also for the determination of the string tension, which is sensitive to the specific form of the fit used to analyze the heavy quark potential at distance $R \simeq (0.25 - 1)$ fm. Such ambiguities are likely to be the origin for the currently existing discrepancy between the calculations presented here and in [18]. However, also the thermodynamic calculations are still sensitive to infra-red effects. While the finite volume effects are quite well under control for the determination of the critical temperature they certainly have to be analyzed in more detail for the discontinuities (surface tension, latent heat) at T_c.

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